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THE EXPANSION OF A FUNCTION INTO A
FOURIER SERIES WITH PRESCRIBED PHASES,
VALID IN THE HALF-PERIOD INTERVAL

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§1. Introduction

A real function of a real variable x may be expanded in the interval $(0, \pi)$ under fairly general conditions either into a cosine series or in a sine series

$$f(x) = \sum_0^{\infty} \alpha_n \cos nx, \quad f(x) = \sum_1^{\infty} \beta_n \sin nx. \quad 1-1$$

In the first case a constant term α_0 is needed, in the second case this is not necessary, and we may expand unity into a sine series

$$1 = \frac{4}{\pi} \left(\frac{\sin x}{1} - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \dots \right), \quad 0 < x < \pi \quad 1-2$$

We shall now investigate the possibility of an a trigonometric expansion of $f(x)$ of the following type

$$f(x) = \sum_1^{\infty} a_n (\sin nx + \gamma_n \cos nx), \quad 0 < x < \pi, \quad 1-3$$

where γ_n may be complex.

This may be considered as a Fourier expansion with prescribed phases valid only in the half-period interval $(0, \pi)$. Sometimes we shall write for 1-3

$$f(x) = \sum_1^{\infty} b_n \sin (nx + \mu_n \pi), \quad 0 < x < \pi, \quad 1-4$$

where

$$\operatorname{tg} \mu_n \pi = \gamma_n, \quad |\operatorname{Re} \mu_n| \leq \frac{1}{2} \quad \text{and} \quad b_n \cos \mu_n \pi = a_n. \quad 1-5$$

In the following sections the case of a positive and real $\gamma_n = \gamma$ which is independent of n is considered first. It appears that generally the expansion 1-3 is always possible and that the coefficients a_n are of order $n^{-1+2\mu}$, where $\mu_n = \mu$ for all n , so that the series 1-3 converges conditionally. In the case $f(x) \equiv 1$ we find in particular

$$1 = \sum_1^{\infty} (-1)^{n-1} e_n(2\mu) \left(\cos nx + \frac{1}{\gamma} \sin nx \right), \quad 1-6$$

where

$$\sum_1^{\infty} e_n(2\mu) \lambda^n = \left(\frac{1+\lambda}{1-\lambda} \right)^{2\mu-1}. \quad 1-7$$

It appears further that a function exists which is orthogonal to all functions $\sin(nx + \mu\pi)$, and we have found

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \sin(nx + \mu\pi) dx = 0, \quad n \geq 1. \quad 1-8$$

If $f(x)$ satisfies the condition

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} f(x) dx = 0 \quad 1-9$$

the series 1-3 converges uniformly and $\sum |a_n|$ and $\sum n|a_n|^2$ are finite. Also the inverse is true. If e.g. $\sum n|a_n|^2 < \infty$ the condition 1-9 is satisfied. If this is the case the coefficients a_n appear to be of the order $n^{-1-2\mu}$.

If $\gamma_n = \gamma$ is a complex constant with $\operatorname{Re} \gamma > 0$ similar conclusions may be drawn since also $0 < \operatorname{Re} \mu < \frac{1}{2}$. If $\operatorname{Re} \gamma < 0$ we may perform the transformation $x \rightarrow \pi - x$, $\gamma \rightarrow -\gamma$ by means of which this case is reduced to the previous one.

If γ is purely imaginary with $|\gamma| > 1$ we have $\operatorname{Re} \mu = \frac{1}{2}$ and the coefficients b_n of 1-4 in general do not tend to zero, so that in this case a convergent expansion of type 1-4, i.e. without constant term, does not exist. If, however, the condition 1-9 is satisfied the series 1-4 converges and $b_n = O(\frac{1}{n})$.

If γ is purely imaginary with $|\gamma| < 1$ we have $\operatorname{Re} \mu = 0$ and the expansion 1-4 is possible with $b_n = O(\frac{1}{n})$ without subsidiary conditions 1-9. However, in this case the orthogonality relation 1-8 breaks down and should be replaced by the two relations 1-10 the first of which may be considered as the analytic continuation of 1-8.

$$\begin{cases} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \{ \sin(nx + \mu\pi) - \sin(n\pi + \mu\pi) \} dx = (-1)^{n-1} \pi, \\ \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} \{ \sin(nx + \mu\pi) - \sin \mu\pi \} dx = \pi. \end{cases} \quad 1-10$$

Thus, there is no function orthogonal to all $\sin(nx + \mu\pi)$, $n \geq 1$, in the ordinary sense.

If again we require that $\sum n|a_n|^2 < \infty$ two conditions are imposed upon $f(x)$

$$\begin{cases} \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \{ f(\pi) - f(x) \} dx = \frac{f(\pi)}{\sin \mu\pi}, \\ \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1+2\mu} \{ f(x) - f(0) \} dx = \frac{f(0)}{\sin \mu\pi}. \end{cases} \quad 1-11$$

Next we shall consider the case of variable γ_n . For simplicity we shall assume that γ_n is real and

$$\gamma_n = \gamma + r_n$$

where $0 < \gamma < \infty$ and $r_n = O(\frac{1}{n^2})$.

We introduce the auxiliary function

$$s(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

from which the coefficients a_n can be determined easily. It appears that the problem of expanding $f(x)$ into a trigonometrical series 1-3 is equivalent to the problem of solving a Cauchy singular integral equation for $s(x)$. This singular integration may be reduced to an ordinary Fredholm equation with bounded kernel

$$F(x) = G(x) - \frac{\cos \mu \pi}{\pi} \int_0^{\pi} K(t, x) F(t) dt, \quad 1-12$$

where

$$F(x) = (1 + \cos x)^{\mu} s(x).$$

$G(x)$ is a known function dependent on $f(x)$ only, and $K(t, x)$ is uniformly bounded in the square $0 \leq t, x \leq \pi$.

If γ_n is a constant, $K(t, x)$ vanishes identically and 1-12 reduces to an explicit expression for $s(x)$ and hence for the coefficients a_n .

If $f(x) \equiv 1$ is developed into an adjoint series

$$1 = \sum_{n=1}^{\infty} h_n (\sin nx + \gamma_n^{-1} \cos nx),$$

the function $h(x)$ defined by

$$h(x) = \sum_{n=1}^{\infty} h_n \sin nx$$

is orthogonal to all functions $\sin nx + \gamma_n \cos nx$.

The orthogonal function may be determined from the following ordinary Fredholm equation

$$H(x) = H_0(x) + \frac{\sin \mu \pi}{\pi} \int_0^{\pi} L(t, x) H(t) dt, \quad 1-13$$

where

$$H(x) = (1 + \cos x)^{\frac{1}{2} - \mu} h(x),$$

$$H_0(x) = (1 - \cos x)^{\frac{1}{2} - \mu} \frac{\sin \mu \pi}{\pi} \int_0^{\pi} h(t) dt.$$

and $L(t, x)$ is a uniformly bounded kernel.

If $\gamma_n = \gamma$ constant, $L(t, x)$ vanishes identically and 1-13 gives merely $h(x) = C(\operatorname{tg} \frac{1}{2}x)^{1-2\mu}$, where C is a constant, in accordance with 1-9.

If γ_n is complex similar conclusions may be drawn since the principle of analytic continuation may be applied. If in particular γ_n is

purely imaginary, the integral equations 1-12 and 1-13 remain solvable.

Finally, a twodimensional potential problem is considered the solution of which may depend upon the expansion of a given function into a trigonometrical series in the half-period interval. By means of Green's theorem the potential problem may be reduced to a singular integral equation, which is another way of showing that the expansion problem 1-3 is equivalent to a singular integral equation.

This report will be followed by another report in which special attention is given to the integral equations 1-12 and 1-13, and to the numerical determination of the orthogonal function $h(x)$.

§2 The Cauchy integral ¹⁾

We need a few properties of Cauchy integrals

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t)dt}{t-z} \quad 2-1$$

where $L=L(a,b)$ is an interval on the real axis with two endpoints a and b , t any point of L , z any point of the complex $z(x,y)$ plane.

If $\varphi(t)$ is R-integrable on L , bounded on L with the possible exception of the endpoints and if $\int_L |\varphi(t)|dt < \infty$ the Cauchy integral $\Phi(z)$ is a holomorphic function in the entire region excluding L , and for large $|z|$

$$\Phi(z) = O\left(\frac{1}{|z|}\right).$$

On the interval L we have the two limits

$$\begin{cases} \Phi^+(x) = \lim_{y \downarrow 0} \Phi(x + yi) \\ \Phi^-(x) = \lim_{y \uparrow 0} \Phi(x + yi) \end{cases}$$

The integral

$$\int_a^b \frac{\varphi(t)dt}{t-x},$$

where $a < x < b$ does not exist in the ordinary sense, but if $\varphi(t)$ satisfies a Hölder condition on L

$$|\varphi(t_2) - \varphi(t_1)| \leq A |t_2 - t_1|^\alpha, \quad \alpha > 0$$

the following limit exists

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_a^{x-\epsilon} \frac{\varphi(t)dt}{t-x} + \int_{x+\epsilon}^b \frac{\varphi(t)dt}{t-x} \right\},$$

1) Cf Muskhelishvili. Singular Integral Equations Ch.2.

and will be written as

$$\int_a^b \frac{\varphi(t)dt}{t-x}.$$

Plemelj ¹⁾ gave the following formulae which are fundamental

$$\Phi^+(x) - \Phi^-(x) = \varphi(x), \quad 2-2$$

$$\Phi^+(x) + \Phi^-(x) = \frac{1}{\pi i} \int_L \frac{\varphi(t)dt}{t-x}. \quad 2-3$$

We shall now calculate some Cauchy integrals which will be needed later on.

Some elementary integrals are given first

$$\int_{-1}^1 (1-t)^\alpha (1+t)^\beta dt = \frac{2^{\alpha+\beta+1} \Gamma(1+\alpha) \Gamma(1+\beta)}{(1+\alpha+\beta)}, \quad 2-4$$

$$\int_{-1}^1 \left(\frac{1-t}{1+t}\right)^\alpha \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{\cos \alpha \pi}, \quad |\operatorname{Re} \alpha| < \frac{1}{2}, \quad 2-5$$

$$\int_0^\pi (\operatorname{tg} \frac{1}{2} t)^{2\alpha} dt = \frac{\pi}{\cos \alpha \pi}, \quad |\operatorname{Re} \alpha| < \frac{1}{2}. \quad 2-6$$

It can be shown easily that

$$\int_{-1}^1 (1-t)^{-\frac{1}{2}+\alpha} (1+t)^{-\frac{1}{2}-\alpha} \frac{dt}{t-z} = -\frac{\pi}{\cos \alpha \pi} \left(\frac{z-1}{z+1}\right)^\alpha \frac{1}{\sqrt{z^2-1}}, \quad |\operatorname{Re} \alpha| < 1. \quad 2-7$$

Proof.

Consider the integral $F(w) = \frac{1}{2\pi i} \int_C (w-1)^{-\frac{1}{2}+\alpha} (w+1)^{-\frac{1}{2}-\alpha} \frac{dw}{w-z}$ where C is a closed contour round the section $(-1,1)$, taken in positive sense, and such that z lies outside C . From the calculus of residues

$$F(w) = -(z-1)^{-\frac{1}{2}+\alpha} (z+1)^{-\frac{1}{2}-\alpha},$$

but on the other hand

$$F(w) = \frac{1}{2\pi} (e^{\alpha \pi i} + e^{-\alpha \pi i}) \int_{-1}^1 (1-t)^{-\frac{1}{2}+\alpha} (1+t)^{-\frac{1}{2}-\alpha} \frac{dt}{t-z}.$$

1) l.c. § 17.

If Plemelj's formula 2-3 is applied to 2-7 we obtain

$$\oint_{-1}^1 (1-t)^{-\frac{1}{2}+\alpha} (1+t)^{-\frac{1}{2}-\alpha} \frac{dt}{t-x} = -\pi \operatorname{tg} \alpha \pi \left(\frac{1-x}{1+x}\right)^{\alpha} \frac{1}{\sqrt{1-x^2}}, \quad |\operatorname{Re} \alpha| < \frac{1}{2} \quad 2-8$$

If in 2-7 and 2-8 we put $t \rightarrow -\cos t$ we obtain

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}t)^{-2\alpha} \frac{dt}{\cos t-z} = -\frac{\pi}{\cos \alpha \pi} \left(\frac{z+1}{z-1}\right)^{\alpha} \frac{1}{\sqrt{z^2-1}}, \quad |\operatorname{Re} \alpha| < \frac{1}{2}, \quad 2-9$$

$$\sin x \oint_0^{\pi} (\operatorname{tg} \frac{1}{2}t)^{-2\alpha} \frac{dt}{\cos t-\cos x} = \pi \operatorname{tg} \alpha \pi (\operatorname{tg} \frac{1}{2}x)^{-2\alpha}, \quad |\operatorname{Re} \alpha| < \frac{1}{2}. \quad 2-10$$

If in 2-7 we put

$$\frac{1}{t-z} = \frac{1}{1-z} + \frac{1-t}{(1-z)(t-z)}$$

we obtain, replacing α by $\alpha - \frac{1}{2}$

$$\int_{-1}^1 (1-t)^{\alpha} (1+t)^{-\alpha} \frac{dt}{t-z} = \frac{\pi}{\sin \alpha \pi} \left\{ \left(\frac{z-1}{z+1}\right)^{\alpha} - 1 \right\}, \quad |\operatorname{Re} \alpha| < 1 \quad 2-11$$

and similarly

$$\oint_{-1}^1 (1-t)^{\alpha} (1+t)^{-\alpha} \frac{dt}{t-x} = \frac{\pi}{\sin \alpha \pi} \left\{ \cos \alpha \pi \left(\frac{1-x}{1+x}\right)^{\alpha} - 1 \right\}, \quad |\operatorname{Re} \alpha| < 1 \quad 2-12$$

From the Cauchy integrals obtained above the following pair easily follows

$$\oint_{-1}^1 \frac{\sqrt{1-t^2}}{t-x} dt = -\pi x, \quad -1 < x < 1, \quad 2-13$$

$$\oint_{-1}^1 \frac{1}{t-x} \frac{dt}{\sqrt{1-t^2}} = 0, \quad -1 < x < 1. \quad 2-14$$

and also

$$\int_{-1}^1 \frac{\sqrt{1-t^2}}{t-z} dt = -\pi (z - \sqrt{z^2-1}), \quad 2-15$$

$$\int_{-1}^1 \frac{1}{t-z} \frac{dt}{\sqrt{1-t^2}} = -\frac{\pi}{\sqrt{z^2-1}}. \quad 2-16$$

A trigonometrical substitution gives

$$\oint_0^{\pi} \frac{\sin^2 t \, dt}{\cos t - \cos x} = -\pi \cos x, \quad 0 < x < \pi, \quad 2-17$$

$$\oint_0^{\pi} \frac{dt}{\cos t - \cos x} = 0, \quad 0 < x < \pi. \quad 2-18$$

A repeated Cauchy integral may be inverted by means of the formula of Poincaré-Bertrand for the proof of which we refer to Muskhelishvili's book ¹⁾.

$$\oint_{-1}^1 \frac{dt}{t-x} \oint_{-1}^1 \frac{\varphi(t,u)du}{t-u} = \pi^2 \varphi(x,x) + \oint_{-1}^1 du \oint_{-1}^1 \frac{\varphi(t,u)dt}{(t-x)(t-u)}. \quad 2-19$$

A trigonometrical substitution gives

$$\oint_0^{\pi} \frac{\sin^2 t \, dt}{\cos t - \cos x} \oint_0^{\pi} \frac{\psi(t,u)du}{\cos t - \cos u} = \pi^2 \psi(x,x) + \oint_0^{\pi} du \oint_0^{\pi} \frac{\sin^2 t \, \psi(t,u)dt}{(\cos t - \cos x)(\cos t - \cos u)} \quad 2-20$$

If in 2-20 $\psi(t,u)$ does not depend on t the right-hand side may be simplified by means of 2-17. We obtain

$$\oint_0^{\pi} \frac{\sin^2 t \, dt}{\cos t - \cos x} \oint_0^{\pi} \frac{\psi(u)du}{\cos t - \cos u} = \pi^2 \psi(u) - \pi \int_0^{\pi} \psi(u)du. \quad 2-21$$

§3 γ a positive real constant

We shall consider the convergent trigonometrical series

$$f(x) = \sum_1^{\infty} a_n (\sin nx + \gamma \cos nx), \quad 0 < x < \pi, \quad 3-1$$

or in equivalent notation

$$f(x) = \sum_1^{\infty} b_n \sin(nx + \mu\pi), \quad 0 < x < \pi, \quad 3-2$$

where

$$a_n = b_n \cos \mu\pi \quad \text{and} \quad \gamma = \operatorname{tg} \mu\pi \quad \text{with} \quad 0 < \mu < \frac{1}{2}.$$

We introduce the analytic function

$$\Phi_0(z) = \sum_1^{\infty} b_n e^{nzi}, \quad z = x+yi, \quad y > 0. \quad 3-3$$

This function satisfies the following conditions

$$x = 0, \pi \quad \operatorname{Im} \Phi_0 = 0, \quad 3-4$$

$$y \rightarrow \infty \quad \Phi_0 \rightarrow 0. \quad 3-5$$

1) l.c. § 23.

The expansion 3-2 of $f(x)$ gives

$$y=0 \quad \text{Im } e^{\mu\pi i} \Phi_0 = f(x), \quad 0 < x < \pi. \quad 3-6$$

In order that 3-2 converges for all x in $(0, \pi)$ we should also have

$$y=0 \quad \left| \int_0^\pi \Phi_0(z) dz \right| < \infty. \quad 3-7$$

We shall see that, given a function $f(x)$ which satisfies a Hölder condition on $(0, \pi)$, the function $\Phi(z)$ is uniquely determined by the conditions 3-4 3-5 3-6 and 3-7. Thus any function of the Hölder type may be represented by a trigonometric series 3-1 or 3-2 and the coefficients a_n or b_n may be found by expanding $\Phi(z)$ in powers of e^{zi} .

The halfstrip $0 < x < \pi \quad y > 0$ may be mapped upon the upper half-plane by means of the mapping function

$$w = -\cos z. \quad 3-8$$

We note the following relations which easily follow from 3-8

$$e^{iz} = -w + \sqrt{w^2 - 1},$$

$$w-1 = -\frac{1}{2} e^{-iz} (1+e^{iz})^2, \quad w+1 = -\frac{1}{2} e^{-iz} (1-e^{iz})^2. \quad 3-9$$

If we write $w = u+vi$ the conditions 3-4 3-5 3-6 become

$$|u| > 1 \quad v=0 \quad \text{Im } \Phi = 0, \quad 3-10$$

$$w \rightarrow \infty \quad \Phi \rightarrow 0, \quad 3-11$$

$$|u| < 1 \quad v=0 \quad \text{Im } e^{\mu\pi i} \Phi = \varphi(u), \quad 3-12$$

$$\text{where} \quad \varphi(-\cos x) = f(x), \quad 3-13$$

$$\text{and} \quad \Phi(-\cos z) = \Phi_0(z).$$

If $\varphi(u) = 0$ the solution of 3-10 and 3-12 is a linear combination of functions

$$X(w) = \left(\frac{w+1}{w-1}\right)^\mu (w+1)^{-\nu_1} (w-1)^{-\nu_2}, \quad 3-14$$

where ν_1 and ν_2 are integers. The condition 3-11 gives $\nu_1 + \nu_2 \geq 1$.

In the z -plane we have near $z=0$ and $z=\pi$

$$X(-\cos z) \sim C_1 (1-e^{iz})^{2\mu-2\nu_1}, \quad z \sim 0,$$

$$X(-\cos z) \sim C_2 (1+e^{iz})^{-2\mu-2\nu_2}, \quad z \sim \pi,$$

where C_1 and C_2 are constant factors.

The condition of integrability 3-7 requires

$$\nu_1 < \mu + \frac{1}{2} \quad \text{and} \quad \nu_2 < -\mu + \frac{1}{2},$$

so that $\nu_1 + \nu_2 < 1$ in contradiction to the result found above.

This shows that, if $f(x) \equiv 0$, the problem considered above has only the zero solution $\Phi = 0$.

This also implies that it is impossible to represent zero by a convergent trigonometric series 3-1. Of course zero may be represented by a trigonometric series 3-1 which is divergent but Abel-summable. Every function $X(w)$ from 3-14 with $\nu_1 + \nu_2 \geq 1$ defines an A-summable trigonometric series.

A particular solution of 3-10, 3-11, 3-12 may be obtained by a Cauchy integral

$$\Phi(w) = \left(\frac{w+1}{w-1}\right)^\mu \frac{1}{\pi} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^\mu \frac{\varphi(t)}{t-w} dt, \quad 3-15$$

for we have

$$e^{i\mu\pi} \Phi = 2i \left(\frac{1+u}{1-u}\right)^\mu \left\{ \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^\mu \frac{\varphi(t)}{t-w} dt \right\}_{w=u+i0},$$

and by Plemelj's formulae 2-2 and 2-3

$$\operatorname{Re} \left\{ \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^\mu \frac{\varphi(t)}{t-w} dt \right\}_{w=u+i0} = \frac{1}{2} \left(\frac{1-u}{1+u}\right)^\mu \varphi(u).$$

Since the problem with $\varphi(u) \equiv 0$ has the zero solution only, the solution 3-15 is unique. We shall show that near $w = \pm 1$

$$\Phi(w) \sim \text{constant}, \quad w \sim -1,$$

$$\Phi(w) \sim (w-1)^{-\mu}, \quad w \sim 1,$$

so that the condition 3-7 also is satisfied.

The behaviour of $\Phi(w)$ near $w=1$ may be determined as follows.

The identity

$$\frac{1-t}{t-w} \varphi(t) = \frac{1-w}{t-w} \varphi(w) - \varphi(t) + \frac{1-w}{t-w} \{ \varphi(t) - \varphi(w) \}, \quad 3-16$$

is substituted into 3-15. We obtain

$$\begin{aligned} \Phi(w) = & - \left(\frac{w+1}{w-1}\right)^\mu \frac{1}{\pi} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \varphi(t) dt + \\ & + (w-1)^{1-\mu} (w+1)^\mu \varphi(w) \frac{1}{\pi} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \frac{dt}{t-w} + \\ & - (w-1)^{1-\mu} (w+1)^\mu \frac{1}{\pi} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \frac{\varphi(t) - \varphi(w)}{t-w} dt. \end{aligned}$$

The second term on the right-hand side may be evaluated by means of 2-7. If $f(x)$ at $x=0$ and $x=\pi$ satisfies a Lipschitz condition of order ϵ ($\epsilon > 0$) $\varphi(t)$ satisfies a Lipschitz condition of order $\frac{1}{2}\epsilon$ at the endpoints ± 1 . Thus the third term is $O(|w-1|^\alpha)$ where $\alpha = \text{Min}(1-\mu, \frac{1}{2}\epsilon)$, ¹⁾ and we may write

$$w \sim 1, \quad \Phi(w) = -A \left(\frac{w+1}{w-1}\right)^\mu + \frac{\varphi(w)}{\sin \mu \pi} + O(|w-1|^\alpha), \quad 3-17$$

where

$$A = \frac{1}{\pi} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \varphi(t) dt. \quad 3-18$$

A similar result may be obtained at $w=-1$. Starting from the identity

$$\frac{\varphi(t)}{t-w} = \frac{\varphi(-1)}{t-w} + \frac{\varphi(t)-\varphi(-1)}{t+1} + \frac{w+1}{t-w} \frac{\varphi(t)-\varphi(-1)}{t+1} \quad 3-19$$

we obtain in view of 2-11

$$w \sim -1, \quad \Phi(w) = \frac{\varphi(-1)}{\sin \mu \pi} + B \left(\frac{w+1}{w-1}\right)^\mu + O(|w+1|^\beta), \quad 3-20$$

where

$$B = \frac{1}{\pi} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^\mu \frac{\varphi(t)-\varphi(-1)}{t+1} dt - \frac{\varphi(-1)}{\sin \mu \pi} \quad 3-21$$

and

$$\beta = \text{Min}(1+\mu, \frac{1}{2}\epsilon).$$

If $\varphi(t)$ is an analytic function or if, at least, $\varphi(t)$ is differentiable a sufficient number of times, the formulae 3-17 and 3-20 may be continued into an expansion of $\Phi(w)$ according to ascending powers of $w \mp 1$ near ± 1 .

We need the following lemma.

Lemma 3 If for every integer $k \geq 0$ $\psi_k(t)$ is defined by

$${}_t\psi_{k+1} \stackrel{\text{def}}{=} \psi_k - \frac{\psi^{(k)}(0)}{k!}, \quad k \geq 1, \quad 3-22$$

$$\psi_0 \stackrel{\text{def}}{=} \psi(t),$$

then we have the following identity

$$\frac{\psi(t)-\psi(w)}{t-w} = \sum_0^{N-1} w^k \psi_{k+1}(t) + \frac{w^N}{t-w} \{ \psi_N(t) - \psi_N(w) \}. \quad 3-23$$

1) Cf Muskhelishvili l.c. § 29.

Proof

Either by means of induction from N to $N+1$ or as follows.

From 3-22 we get

$$t^k \psi_k = \psi - \sum_{j=0}^{k-1} c_j t^j, \quad k \geq 1,$$

where $c_j = \frac{\psi^{(j)}(0)}{j!}$. Then we have the following reduction.

$$\begin{aligned} \sum_{k=0}^{N-1} w^k \psi_{k+1}(t) &= \psi(t) \sum_{k=0}^{N-1} \frac{w^k}{t^{k+1}} - \sum_{k=0}^{N-1} \frac{w^k}{t^{k+1}} \sum_{j=0}^k c_j t^j = \\ &= \left(1 - \frac{w^N}{t^N}\right) \frac{\psi(t)}{t-w} - \sum_{j=0}^{N-1} c_j t^j \sum_{k=j}^{N-1} \frac{w^k}{t^{k+1}} = \\ &= \left(1 - \frac{w^N}{t^N}\right) \frac{\psi(t)}{t-w} - \frac{1}{t-w} \sum_{j=0}^{N-1} c_j w^j + \frac{1}{t-w} \frac{w^N}{t^N} \sum_{j=0}^{N-1} c_j t^j = \\ &= \frac{\psi(t) - \psi(w)}{t-w} - w^N \frac{\psi_N(t) - \psi_N(w)}{t-w}. \end{aligned}$$

The identity 3-23 may lead to the formal expansion

$$\frac{\psi(t) - \psi(w)}{t-w} = \sum_{k=0}^{\infty} w^k \psi_{k+1}(t). \quad 3-24$$

The functions $t^k \psi_k(t)$ are the remainders of the Taylor expansion of $\psi(t)$ at $t=0$. A similar expansion may be found for the remainders of the Taylor expansion of $\psi(t)$ at an arbitrary point t_0 if upon 3-24 or 3-23 a linear transformation $t \rightarrow t-t_0$ is applied.

If $t^k \varphi_k(t)$ and $t^k \varphi_k^*(t)$ are the remainders of the Taylor expansion of $\varphi(t)$ at $t=1$ and $t=-1$ respectively we have from 3-24

$$\frac{\varphi(t) - \varphi(w)}{t-w} = \sum_{k=0}^{\infty} (w-1)^k \varphi_{k+1}(t) = \sum_{k=0}^{\infty} (w+1)^k \varphi_{k+1}^*(t). \quad 3-25$$

The behaviour of $\Phi(w)$ at $w=1$ can be determined as follows.

From 3-25 we may derive

$$\frac{1-t}{t-w} \varphi(t) = -\frac{1-w}{t-w} \varphi(w) - \sum_{k=0}^{\infty} (w-1)^k \varphi_k(t),$$

so that

$$\Phi(w) = \frac{\varphi(w)}{\sin \mu \pi} - \left(\frac{w+1}{w-1}\right)^{\mu} \sum_{k=0}^{\infty} A_k (w-1)^k, \quad w \rightarrow 1, \quad 3-26$$

where

$$A_k = \frac{1}{\pi} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \varphi_k(t) dt . \quad 3-27$$

Similarly the expansion

$$\frac{\varphi(t)}{t-w} = \frac{\varphi(w)}{t-w} + \sum_0^{\infty} (w+1)^k \varphi_{k+1}^*(t) ,$$

from 3-25 gives

$$\Phi(w) = \frac{\varphi(w)}{\sin \mu \pi} \left\{ 1 - \left(\frac{w+1}{w-1} \right)^{\mu} \right\} + \left(\frac{w+1}{w-1} \right)^{\mu} \sum_0^{\infty} B_k (w+1)^k , \quad w \rightarrow -1 \quad 3-28$$

where

$$B_k = \frac{1}{\pi} \int_{-1}^1 (1-t)^{\mu} (1+t)^{-\mu} \varphi_{k+1}^*(t) dt . \quad 3-29$$

§4 Special cases

The coefficient of s^n in the expansion of $\left(\frac{1+s}{1-s} \right)^{\alpha}$ will be denoted by $e_n(\alpha)$.

$$\left(\frac{1+s}{1-s} \right)^{\alpha} = \sum_0^{\infty} e_n(\alpha) s^n , \quad |s| < 1 . \quad 4-1$$

We note the following elementary properties

$$e_n(-\alpha) = (-1)^n e_n(\alpha) , \quad 4-2$$

$$n e_n(\alpha) = 2\alpha e_{n-1}(\alpha) + (n-2) e_{n-2}(\alpha) , \quad 4-3$$

$$e_0=1 , \quad e_1=2\alpha , \quad e_2=2\alpha^2 , \quad e_3=\frac{4\alpha^3+2\alpha}{3} \dots ,$$

$$e_n(\alpha+1) = e_n(\alpha) + 2 \sum_0^{n-1} e_k(\alpha) . \quad 4-4$$

If in 4-1 $s=e^{ix}$ is substituted we obtain by separating real and imaginary part

$$\cos \frac{\alpha \pi}{2} (\operatorname{tg} \frac{1}{2} x)^{-\alpha} = \sum_0^{\infty} e_n(\alpha) \cos nx , \quad |\operatorname{Re} \alpha| < 1 . \quad 4-5$$

and

$$\sin \frac{\alpha \pi}{2} (\operatorname{tg} \frac{1}{2} x)^{-\alpha} = \sum_1^{\infty} e_n(\alpha) \sin nx , \quad |\operatorname{Re} \alpha| < 1 . \quad 4-6$$

From 4-5 and 4-6 we obtain for $|\operatorname{Re} \alpha| < 1$

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2} x)^{-\alpha} \sin nx dx = \frac{\pi}{2} \frac{e_n(\alpha)}{\sin \frac{\alpha \pi}{2}} , \quad n > 0 , \quad 4-7$$

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-\alpha} \cos nx \, dx = \frac{\pi}{2} \frac{e_n(\alpha)}{\cos \frac{\alpha\pi}{2}}, \quad n > 0, \quad 4-8$$

Also we have from 4-5 and 4-6

$$1 = - \sum_{n=1}^{\infty} e_n(\alpha) (\cos nx - \cotg \frac{\alpha\pi}{2} \sin nx), \quad 4-9$$

and from 4-7 and 4-8

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-\alpha} (\sin nx - \cotg \frac{\alpha\pi}{2} \cos nx) \, dx = 0 \quad 4-10$$

for all $n \geq 1$.

If α is real and positive the coefficients $e_n(\alpha)$ are positive. We shall prove that

$$e_n(\alpha) \leq \frac{2\alpha}{n^{1-\alpha}} \quad n \geq 1, \quad 0 < \alpha \leq 1. \quad 4-11$$

The assertion is clearly true for $n=1,2$. In order to apply the principle of induction we have to prove the inequality

$$\frac{2\alpha}{(n-1)^{1-\alpha}} + \frac{1}{(n-2)^{-\alpha}} \leq n^{\alpha} \quad n \geq 3$$

Consider the function $V(x) = (1-2x)^{\alpha} + 2\alpha x (1-x)^{\alpha-1}$ for $0 < x < 1/3$. Its Taylor expansion is

$$V(x) = 1 - \binom{\alpha}{3} (2^3-6) x^3 + \binom{\alpha}{4} (2^4-8) x^4 \dots$$

i.e. 1 followed by negative terms. From this we get $V(x) \leq 1$ or $V(\frac{1}{n}) \leq 1$ so that the principle of induction holds.

Next we shall derive an asymptotic estimation for $e_n(\alpha)$ in the case $0 \leq \operatorname{Re} \alpha < 1$.

Starting from

$$e_n(\alpha) = \frac{1}{2\pi i} \oint \left(\frac{1+s}{1-s}\right)^{\alpha} \frac{ds}{s^{n+1}}, \quad 4-12$$

we have by a suitable deformation of the contour

$$\frac{\pi}{\sin \alpha\pi} e_n(\alpha) = \int_1^{\infty} \left(\frac{t+1}{t-1}\right)^{\alpha} \frac{dt}{t^{n+1}} - (-1)^n \int_1^{\infty} \left(\frac{t-1}{t+1}\right)^{\alpha} \frac{dt}{t^{n+1}}. \quad 4-13$$

In the first integral of the right-hand side we substitute $t=e^u$

$$\int_1^{\infty} \left(\frac{t+1}{t-1}\right)^{\alpha} \frac{dt}{t^{n+1}} = \int_0^{\infty} e^{-nu} \left(\frac{e^u+1}{e^u-1}\right)^{\alpha} du.$$

Since

$$\left(\frac{e^u+1}{e^u-1}\right)^{\alpha} = \left(\frac{2}{u}\right)^{\alpha} \{1 + O(u^2)\}$$

the last integral equals $\frac{2^\alpha \Gamma(1-\alpha)}{n^{1-\alpha}} + \mathcal{O}(n^{-3+\alpha})$.

In a similar way the second integral of 4-13 is $\frac{2^{-\alpha} \Gamma(1+\alpha)}{n^{1+\alpha}} + \mathcal{O}(n^{-3-\alpha})$. Thus we have the following asymptotic expression

$$e_n(\alpha) = \frac{2^\alpha}{\Gamma(\alpha) n^{1-\alpha}} + \frac{(-1)^n 2^{-\alpha}}{\Gamma(-\alpha) n^{1+\alpha}} + \mathcal{O}(n^{-3+\alpha}), \quad 0 \leq \operatorname{Re} \alpha < 1.$$

We note the following expression for $e_n(\alpha)$

$$e_n(\alpha) = (-1)^{n-1} 2^\alpha F(1-n, 1+\alpha, 2, 2), \quad n \geq 1, \quad -1 < \operatorname{Re} \alpha < 1.$$

4-14

which may be obtained from 4-12 as follows

$$\begin{aligned} \frac{1}{2\pi i} \oint \left(\frac{1+s}{1-s}\right)^\alpha \frac{ds}{s^{n+1}} &= \frac{\sin \alpha \pi}{\pi} \int_{-1}^1 (1+t)^\alpha (1-t)^{-\alpha} t^{n-1} dt = \\ &= \frac{(-1)^{n-1} 2 \sin \alpha \pi}{\pi} \int_0^1 t^\alpha (1-t)^{-\alpha} (1-2t)^{n-1} dt. \end{aligned}$$

We return now to the original expansion

$$f(x) = \sum_1^\infty a_n (\sin nx + \gamma \cos nx), \quad 4-15$$

or

$$f(x) = \sum_1^\infty b_n \sin (nx + \mu \pi).$$

If we take $f(x) \equiv 1$, whence by 3-13 $\varphi(u)=1$, 3-15 gives by 2-11

$$\sin \mu \pi \Phi(w) = 1 - \left(\frac{w+1}{w-1}\right)^\mu,$$

or (cf 3-13)

$$\sin \mu \pi \Phi_0(z) = 1 - \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\mu},$$

from which the coefficients a_n or b_n follow easily. We obtain

$$1 = \sum_1^\infty (-1)^{n-1} e_n(2\mu) \left(\cos nx + \frac{1}{\gamma} \sin nx\right), \quad 4-16$$

which is identical with 4-9 if $\alpha = -2\mu$.

If we take $f(x) \equiv (\operatorname{tg} \frac{1}{2}x)^{2\sigma}$, 3-15 gives

$$\begin{aligned} \sin (\mu - \sigma) \pi \Phi(w) &= \left(\frac{w+1}{w-1}\right)^\sigma - \left(\frac{w+1}{w-1}\right)^\mu \\ &= \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\sigma} - \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\mu}, \end{aligned}$$

again with $w = -\cos z$, so that the coefficients of

$$(\operatorname{tg} \frac{1}{2}x)^{2\sigma} = \sum_1^{\infty} b_n(\sigma) \sin(nx + \mu\pi) \quad 4-17$$

become

$$b_n(\sigma) = \frac{e_n(-2\sigma) - e_n(-2\mu)}{\sin(\mu - \sigma)\pi} . \quad 4-18$$

The particular case $\sigma = \mu$ gives

$$b_n(\mu) = \frac{-1}{\mu \cos \mu\pi} - \frac{d}{d\mu} e_n(-2\mu) , \quad 4-19$$

with the asymptotic estimate

$$b_n(\mu) \sim \frac{(-1)^{n-1} 2^{2\mu}}{\pi \Gamma(2\mu-1)} \frac{\ln n}{n^{1-2\mu}} . \quad 4-20$$

§5 The expansion of $f(x)$

The coefficients of the expansion of a function $f(x)$ which satisfies a Hölder condition on $(0, \pi)$ are determined by the expansion of $\Phi_0(z)$ from 3-3 in powers of e^{zi} .

If again

$$f(x) = \sum_1^{\infty} b_n \sin(nx + \mu\pi), \quad 5-1$$

we have

$$b_n = \frac{1}{2\pi i} \oint \frac{\Phi_1(s)}{s^{n+1}} ds , \quad 5-2$$

where

$$\Phi_1(e^{zi}) = \Phi_0(z).$$

We know that $\Phi(w)$ from 3-15 is holomorphic in the entire w -region except at $w=\pm 1$. The asymptotic behaviour of b_n for large n is determined by the singularities of $\Phi_1(s)$ at $s=\pm 1$ which correspond to those of $\Phi(w)$ at $w=\pm 1$ by means of the conformal transformation

$$w = -\frac{1}{2} \left(s + \frac{1}{s} \right) , \quad s = -w + \sqrt{w^2 - 1}. \quad 5-3$$

The transformation 5-3 maps the w -plane upon the interior of the unit circle in the s -plane. Circles $|s| = \text{constant}$ correspond to ellipses with foci ± 1 in the w -plane. The section $(-1, 1)$ in the w -plane corresponds to the unit-circle in the s -plane.

The behaviour at the singularities $s = \pm 1$ follows from 3-17 and 3-20. If $f(x)$ at $x=0$ and $x=\pi$ satisfies a Hölder condition of order ε ($\varepsilon > 0$) we have

$$s \rightarrow 1 \quad \Phi_1(s) = \frac{f(0)}{\sin \mu\pi} + B \left(\frac{1-s}{1+s} \right)^{2\mu} + \mathcal{O}(|1-s|^{2\beta}) , \quad 5-4$$

$$s \rightarrow -1 \quad \Phi_2(s) = -A \left(\frac{1-s}{1+s} \right)^{2\mu} + \frac{f(\pi)}{\sin \mu \pi} + \mathcal{O}(|1+s|^{-2\alpha}), \quad 5-5$$

where A and B are given by 3-18 and 3-21, or

$$A = \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} f(x) dx, \quad 5-6$$

$$B = \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} \{ f(x) - f(0) \} dx - \frac{f(0)}{\sin \mu \pi}, \quad 5-7$$

and $\alpha = \operatorname{Min} (1-\mu, \frac{1}{2}\varepsilon),$

$$\beta = \operatorname{Min} (1+\mu, \frac{1}{2}\varepsilon).$$

By a similar procedure as in the previous section we obtain for b_n the following asymptotic estimate

$$b_n = \frac{(-1)^{n+1} 2^{2\mu}}{\Gamma(2\mu)} \frac{A}{n^{1-2\mu}} + \frac{2^{-2\mu}}{\Gamma(-2\mu)} \frac{B}{n^{1+2\mu}} + \mathcal{O}(n^{-1-2\alpha}). \quad 5-8$$

The second term in the right-hand side of 5-8 is significant only if $f(x)$ is hölderian at $x=\pi$ of an order which exceeds 2μ . This is true e.g. if $f(x)$ is differentiable at $x=\pi$.

§6 γ is a complex constant

We shall now consider the expansion

$$f(x) = \sum_{n=1}^{\infty} a_n (\sin nx + \gamma \cos nx), \quad 0 \leq x \leq \pi, \quad 6-1$$

where γ is complex.

We put

$$\gamma = \operatorname{tg} \mu \pi, \quad \mu = \frac{1}{2\pi i} \ln \frac{1+\gamma i}{1-\gamma i}, \quad 6-2$$

and we shall consider the strip $-\frac{1}{2} \leq \operatorname{Re} \mu \leq \frac{1}{2}$.

The right-hand half-plane of γ corresponds to $0 < \operatorname{Re} \mu < \frac{1}{2}$.

The unit-circle $|\gamma|=1$ with the exception of $\gamma = \pm i$ corresponds to $\operatorname{Re} \mu = \pm \frac{1}{4}$.

The imaginary axis of γ with $|\operatorname{Im} \gamma| > 1$ corresponds to $\operatorname{Re} \mu = \pm \frac{1}{2}$.

The imaginary axis of γ with $|\operatorname{Im} \gamma| < 1$ corresponds to $\operatorname{Re} \mu = 0$.

The results obtained in the preceding sections for γ real and positive may be extended easily to the case of complex γ . Since $\Phi(w)$ from 3-15 is an analytic function of μ the principle of analytic contin-

uation may be applied throughout.

We shall now consider the main result, the asymptotic estimation of a_n (or b_n) from formula 5-8. The coefficients A and B are given by 3-18 and 3-21. The expression for B is valid for the entire region $|\operatorname{Re} \mu| \leq \frac{1}{2}$ whereas the expression for A is only valid for $0 < \operatorname{Re} \mu \leq \frac{1}{2}$. However, an analytic continuation may easily be obtained.

We have for $|\operatorname{Re} \mu| \leq \frac{1}{2}$

$$A = -\frac{1}{\pi} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t) - \varphi(1)}{t-1} dt + \frac{\varphi(1)}{\sin \mu \pi},$$

$$B = \frac{1}{\pi} \int_{-1}^1 \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t) - \varphi(-1)}{t+1} dt - \frac{\varphi(-1)}{\sin \mu \pi},$$

or

$$A = \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \{f(x) - f(\pi)\} dx + \frac{f(\pi)}{\sin \mu \pi}, \quad 6-3$$

$$B = \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} \{f(x) - f(0)\} dx - \frac{f(0)}{\sin \mu \pi}. \quad 6-4$$

For simplicity the following conclusions will be founded upon the assumption that $f(x)$ is differentiable in $0 \leq x \leq \pi$.

a $0 < \operatorname{Re} \mu < \frac{1}{2}$.

The leading term of the asymptotic expansion of a_n is proportional to $A n^{-1+2\mu}$ so that the following conclusion may be drawn:

The trigonometrical expansion 6-1 is possible and

$$a_n = O(n^{-1+2\operatorname{Re} \mu}).$$

The condition $\sum |a_n| < \infty$ or $\sum n |a_n|^2 < \infty$ requires $A = 0$ or

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} f(x) dx = 0, \quad 6-5$$

and if $A=0$ we have

$$a_n = O(n^{-1-2\operatorname{Re} \mu}).$$

b $-\frac{1}{2} < \operatorname{Re} \mu < 0$.

This case may be reduced to the previous one by means of the substitution $x \rightarrow \pi - x$, $\mu \rightarrow -\mu$. The condition 6-5 becomes

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} f(x) dx = 0. \quad 6-6$$

c $\operatorname{Re} \mu = 0$.

In this case corresponding to a pure imaginary γ with $|\gamma| \leq 1$ the leading term of the asymptotic expansion contains both $A n^{-1}$ and $B n^{-1}$.

The expansion 6-1 is possible and

$$a_n = O(n^{-1}) .$$

The condition $\sum |a_n| < \infty$ or $\sum n |a_n|^2 < \infty$ requires both $A=0$ and $B=0$, thus from 6-3 and 6-4

$$\begin{cases} \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \{f(x) - f(\pi)\} + \frac{f(\pi)}{\sin \mu \pi} = 0, \\ \frac{1}{\pi} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} \{f(x) - f(0)\} - \frac{f(0)}{\sin \mu \pi} = 0 . \end{cases} \quad 6-7$$

Thus, if and only if the two conditions 6-7 are satisfied we have

$$a_n = O(n^{-2}) .$$

If in 6-1 we take $f(x) = \sin nx + \gamma \cos nx$ one of the conditions 6-5, 6-6, 6-7 should be satisfied and thus another proof is obtained for the following relation of orthogonality.

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} (\sin nx + \gamma \cos nx) dx = 0, \quad n \geq 1, \quad \operatorname{Re} \mu > 0, \quad 6-8$$

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} (\sin nx + \gamma \cos nx) dx = 0, \quad n \geq 1, \quad \operatorname{Re} \mu < 0; \quad 6-9$$

$$\begin{cases} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \{ \sin(nx + \mu\pi) - \sin(n\pi + \mu\pi) \} dx = (-1)^{n-1} \pi, \\ \int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{-1-2\mu} \{ \sin(nx + \mu\pi) - \sin \mu\pi \} dx = \pi . \end{cases} \quad \begin{matrix} n \geq 1, \operatorname{Re} \mu = 0 \\ n \geq 1, \operatorname{Re} \mu = 0 \end{matrix} \quad 6-10$$

d $\operatorname{Re} \mu = \frac{1}{2}$.

In this case, and only here, a convergent expansion 6-1 is not possible unless $A=0$.

If $A \neq 0$ we have $a_n = O(1)$ and $\lim_{n \rightarrow \infty} a_n$ does not exist.

If $A=0$ we have $a_n = O(n^{-2})$.

However, in this case the relation of orthogonality 6-8 remains true.

Taking up the general case $0 < \operatorname{Re} \mu < \frac{1}{2}$ we consider an expansion which also includes a constant term

$$f(x) = a'_0 + \sum_1^{\infty} a'_n (\sin nx + \gamma_n \cos nx) \quad 6-11$$

If a convergent expansion 6-1 without constant term exists the convergence may be improved by adding an appropriate constant. If a'_0 is chosen in such a way that for the function $f(x)-a'_0$ the condition 6-5 is fulfilled we have

$$a'_0 = A \sin \mu \pi \quad a'_n = \mathcal{O}(n^{-1-2\operatorname{Re}\mu})$$

and
$$a'_n = a_n + (-1)^n \cos \mu \pi e_n(2\mu) A$$

If $\operatorname{Re}\mu = \frac{1}{2}$ the expansion 6-1 is divergent but 6-11 converges.

If $\operatorname{Re}\mu = 0$ nothing is gained since both a_n and a'_n are $\mathcal{O}(n^{-1})$.

§7 γ_n is variable

We consider the general problem of the following expansion

$$f(x) = \sum_1^{\infty} a_n (\sin nx + \gamma_n \cos nx), \quad 0 < x < \pi, \quad 7-1$$

where $f(x)$ satisfies a Hölder condition in the interval $0 \leq x \leq \pi$, and where γ_n is a function of n of the following type

$$\gamma_n = \gamma + r_n, \quad r_n = \mathcal{O}(n^{-2}), \quad \operatorname{Re} \gamma > 0. \quad 7-2$$

If we introduce the auxiliary functions

$$s(x) = \sum_1^{\infty} a_n \sin nx, \quad 7-3$$

$$c(x) = \sum_1^{\infty} a_n \gamma_n \cos nx, \quad 7-4$$

we have

$$f(x) = s(x) + c(x), \quad 7-5$$

and for a_n we have

$$\frac{\pi}{2} a_n = \int_0^{\pi} s(x) \sin nx \, dx = \frac{1}{\gamma_n} \int_0^{\pi} c(x) \cos nx \, dx. \quad 7-6$$

Thus, for all $n \geq 1$ we have the relations

$$\gamma_n \int_0^{\pi} s(x) \sin nx \, dx - \int_0^{\pi} c(x) \cos nx \, dx = 0. \quad 7-7$$

The infinite number of relations 7-7 can be shown to be equivalent to a single integral equation of the Cauchy type. We need a few lemmas.

lemma 7-1
$$\frac{1}{2} + \sum_{n=1}^N \cos nt = \frac{\sin (N+\frac{1}{2})t}{2 \sin \frac{1}{2}t}.$$

$$\sum_{n=1}^N \sin nt = \frac{\cos \frac{1}{2}t - \cos(N+\frac{1}{2})t}{2 \sin \frac{1}{2}t}.$$

Proof Take real and imaginary part of $\sum_{n=1}^N e^{int}$.

lemma 7-2
$$\lim_{N \rightarrow \infty} \int_0^{\pi} \varphi(t) \sum_{n=1}^N \cos n(t-x) dt = \begin{cases} \pi \varphi(x) - \frac{1}{2} \int_0^{\pi} \varphi(t) dt, & \text{if } 0 < x < \pi, \\ -\frac{1}{2} \int_0^{\pi} \varphi(t) dt, & \text{if } -\pi < x < 0. \end{cases}$$

Proof from Riemann's theorem for a Fourier series.

lemma 7-3
$$\lim_{N \rightarrow \infty} \int_0^{\pi} \varphi(t) \sum_{n=1}^N \sin n(t-x) dt = \frac{1}{2} \oint_0^{\pi} \cotg \frac{1}{2}(t-x) \varphi(t) dt,$$

where $\oint$ represents a Cauchy integral if $0 < x < \pi$.

Proof. If $-\pi < x < 0$ the relation follows from Riemann's theorem. The Cauchy integral reduces in this case to a Riemann integral.

If $0 < x < \pi$ consider $\lim_{N \rightarrow \infty} \int_0^{x-\epsilon} + \int_{x+\epsilon}^{\pi}.$

If the relations 7-7 are multiplied either by $\sin nx$ or $\cos nx$ then summation yields an integral equation.

Taking the sine multiplication first we obtain from 7-7

$$\begin{aligned} \int_0^{\pi} s(t) \sum_{n=1}^N \gamma_n \{ \cos n(t-x) - \cos n(t+x) \} dt + \\ + \int_0^{\pi} c(t) \sum_{n=1}^N \{ \sin n(t-x) - \sin n(t+x) \} dt = 0. \end{aligned}$$

For $N \rightarrow \infty$ we find by means of lemma 7-2 and 7-3

$$\gamma s(x) - \frac{1}{\pi} \oint_0^{\pi} \frac{\sin x c(t) dt}{\cos t - \cos x} + \frac{1}{\pi} \int_0^{\pi} K_1(t, x) s(t) dt = 0, \quad 7-8$$

where

$$K_1(t, x) = 2 \sum_{n=1}^{\infty} r_n \sin nt \sin nx. \quad 7-9$$

If the cosine multiplication is taken we obtain in a similar way

$$c(x) + \frac{\gamma}{\pi} \oint_0^{\pi} \frac{\sin t s(t) dt}{\cos t - \cos x} - \frac{1}{\pi} \int_0^{\pi} K_2(t, x) s(t) dt = 0, \quad 7-10$$

where

$$K_2(t, x) = 2 \sum_{n=1}^{\infty} r_n \sin nt \cos nx. \quad 7-11$$

Since K_1 and K_2 are of the form

$$K_1 = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad K_2 = \sum_{n=1}^{\infty} \alpha_n \cos nx,$$

$K_2(x)$ is the series conjugate to $K_1(x)$, and hence

$$K_2(x) = \frac{1}{2\pi} \int_0^{\pi} \cotg \frac{u}{2} \{ K_1(x+u) - K_1(x-u) \} du.$$

Since $K_1(x)$ is odd this expression may be written in the form

$$K_2(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{\sin u K_1(u) du}{\cos u - \cos x}. \quad 7-12$$

The integral equations 7-8 and 7-10 are not entirely equivalent since the condition

$$\int_0^{\pi} c(x) dx = 0, \quad 7-13$$

expressing the absence of a constant term in the expansion 7-1 is used in the derivation of 7-10 but not in that of 7-8.

Conversely 7-13 follows from 7-10, for, if we integrate from 0 to π the second term vanishes in view of 2-18 and the third term vanishes in

view of $\int_0^{\pi} \cos nx dx = 0$ for each $n \geq 1$.

By means of the formula of Poincaré-Bertrand of § 2 the integral equation 7-10 can be shown to be equivalent to the integral equation 7-8 plus the condition 7-13. If e.g. upon 7-8 the following singular operator Ω is applied

$$\Omega \varphi(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\varphi(t) \sin t dt}{\cos t - \cos x},$$

a repeated Cauchy integral is obtained upon which formula 2-21 may be applied which gives 7-10.

The singular integral equation 7-10 will now be studied more closely according to the methods given by Muskhelishvili ¹⁾. The equation may be written as

1) l.c. part V ch. 14.

$$s(x) - \frac{\gamma}{\pi} \int_0^{\pi} \frac{\sin t \, s(t) \, dt}{\cos t - \cos x} + \frac{1}{\pi} \int_0^{\pi} K_2(t, x) \, s(t) \, dt = f(x) \quad 7-14$$

The dominant equation of 7-14 is

$$s_d(x) - \frac{\gamma}{\pi} \int_0^{\pi} \frac{\sin t \, s_d(t) \, dt}{\cos t - \cos x} = f(x). \quad 7-15$$

If we put

$$\begin{aligned} u &= -\cos x, \\ s_d(x) &= \varphi(u), \\ f(x) &= \psi(u), \end{aligned}$$

7-15 passes into

$$\varphi(u) + \frac{\gamma}{\pi} \int_{-1}^1 \frac{\varphi(t) \, dt}{t-u} = \psi(u). \quad 7-16$$

If $\Phi(w)$ represents the Cauchy integral of $\varphi(u)$

$$\Phi(w) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\varphi(t) \, dt}{t-w}$$

7-16 may be brought in the form

$$(1 + \gamma i) \Phi^+ - (1 - \gamma i) \Phi^- = \psi.$$

This problem is very similar to that considered in section 2 and the solution of 7-16 may be written down immediately (cf formula 3-15)

$$\Phi(w) = \cos \mu \pi \left(\frac{w+1}{w-1} \right)^{\mu} \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{1-t}{1+t} \right)^{\mu} \frac{\psi(t) \, dt}{t-w}. \quad 7-17$$

In fact, we have from Plemelj's formulae 2-2, 2-3

$$\begin{aligned} \Phi^+(u) &= e^{-\mu \pi i} \cos \mu \pi \left(\frac{1+u}{1-u} \right)^{\mu} \left\{ \frac{1}{2} \left(\frac{1-u}{1+u} \right)^{\mu} \psi(u) + \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{1-t}{1+t} \right)^{\mu} \frac{\psi(t) \, dt}{t-u} \right\}, \\ \Phi^-(u) &= e^{\mu \pi i} \cos \mu \pi \left(\frac{1+u}{1-u} \right)^{\mu} \left\{ -\frac{1}{2} \left(\frac{1-u}{1+u} \right)^{\mu} \psi(u) + \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{1-t}{1+t} \right)^{\mu} \frac{\psi(t) \, dt}{t-u} \right\}, \end{aligned}$$

so that, again from 2-2, $\varphi(u)$ becomes

$$\varphi(u) = \cos^2 \mu \pi \psi(u) - \frac{\sin 2\mu \pi}{2\pi} \left(\frac{1+u}{1-u} \right)^{\mu} \int_{-1}^1 \left(\frac{1-t}{1+t} \right)^{\mu} \frac{\psi(t) \, dt}{t-u}, \quad 7-18$$

or for $s(x)$

$$s(x) = \cos^2 \mu \pi f(x) + \frac{\sin 2\mu\pi}{2\pi} (\operatorname{tg} \frac{1}{2}x)^{2\mu} \int_0^\pi (\operatorname{tg} \frac{1}{2}t)^{-2\mu} \frac{\sin t f(t) dt}{\cos t - \cos x}. \quad 7-19$$

If γ_n is constant the function $K_2(t, x)$ vanishes and 7-19 represents the unique solution of the integral equation 7-10¹⁾. If γ_n is variable $f(x)$ in 7-19 may be replaced by

$$f(x) = \frac{1}{\pi} \int_0^\pi K_2(t, x) s(t) dt,$$

and the following integral equation of the Fredholm type is obtained

$$F(x) = G(x) - \frac{\cos \mu\pi}{\pi} \int_0^\pi K(t, x) F(t) dt, \quad 7-20$$

where

$$F(x) = (1 + \cos x)^\mu s(x), \quad 7-21$$

$$G(x) = \cos^2 \mu \pi (1 + \cos x)^\mu f(x) + \frac{\sin 2\mu\pi}{2\pi} (1 - \cos x)^\mu \int_0^\pi \left(\frac{1 + \cos t}{1 - \cos t} \right)^\mu \frac{\sin t f(t) dt}{\cos t - \cos x}, \quad 7-22$$

$$K(t, x) = (1 + \cos t)^{-\mu} (1 - \cos x)^\mu \sum_{n=1}^{\infty} r_n \sin nt k_n(x), \quad 7-23$$

$$k_n(x) = 2 \cos \mu \pi \left(\frac{1 + \cos x}{1 - \cos x} \right)^\mu \cos nx + \frac{2 \sin \mu \pi}{\pi} \int_0^\pi \left(\frac{1 + \cos u}{1 - \cos u} \right)^\mu \frac{\sin u \cos nu du}{\cos u - \cos x}. \quad 7-24$$

We shall use the following lemmas.

lemma 7-4

$$1 + 2 \sum_{n=1}^{\infty} \lambda^n \cos nx = \frac{1 - \lambda^2}{1 - 2\lambda \cos x + \lambda^2}, \quad 7-25$$

$$\sum_{n=1}^{\infty} \lambda^n \sin nx = \frac{\lambda \sin x}{1 - 2\lambda \cos x + \lambda^2}. \quad 7-26$$

Proof

Take real and imaginary part of $\sum_{n=0}^{\infty} (\lambda e^{ix})^n = (1 - \lambda e^{ix})^{-1}$.

lemma 7-5

If $|\theta_n| < 1$, $0 < \alpha < 1$, we have

1) cf §3, 3-15.

$$\left| \sum_{n=1}^{\infty} \frac{\theta_n \sin nx}{n^{1+\alpha}} \right| \leq \frac{5 \sin^{\alpha} x}{4\alpha(1-\alpha)}, \quad 0 < x < \pi \quad 7-27$$

Proof If $N = \left[\frac{1}{\sin x} \right]$ we have $\left| \frac{\theta_{N+1} \sin(N+1)x}{(N+1)^{1+\alpha}} \right| \leq (N+1)^{-1-\alpha} \leq \frac{\sin^{\alpha} x}{4\alpha(1-\alpha)}$

$$\left| \sum_{n=1}^N \frac{\theta_n \sin nx}{n^{1+\alpha}} \right| \leq \sin x \sum_{n=1}^N \frac{1}{n^{\alpha}} \leq \sin x \int_0^N \frac{dt}{t} = \frac{N^{1-\alpha} \sin x}{1-\alpha} \leq \frac{\sin^{\alpha} x}{1-\alpha},$$

$$\left| \sum_{n=N+2}^{\infty} \frac{\theta_n \sin nx}{n^{1+\alpha}} \right| \leq \sum_{n=N+2}^{\infty} \frac{1}{n^{1+\alpha}} \leq \int_{N+1}^{\infty} \frac{dt}{t^{1+\alpha}} = \frac{(N+1)^{-\alpha}}{\alpha} \leq \frac{\sin^{\alpha} x}{\alpha}.$$

The expression 7-24 for $k_n(x)$ can be simplified considerably. If the following generating function is introduced

$$k(x, \lambda) = \frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \lambda^n, \quad 1, \quad 7-28$$

we obtain by means of 7-25

$$(1-\lambda^2)^{-1} k(x, \lambda) = \frac{\cos \mu\pi (\operatorname{tg} \frac{1}{2}x)^{-2\mu}}{1-2\lambda \cos x + \lambda^2} + \frac{\sin \mu\pi}{\pi} \int_0^{\pi} \left(\frac{1+\cos u}{1-\cos u} \right)^{\mu} \frac{\sin u du}{(1-2\lambda \cos u + \lambda^2)(\cos u - \cos x)}.$$

The second term in the right-hand side may be written as

$$\frac{\sin \mu\pi}{\pi} \frac{1}{1-2\lambda \cos x + \lambda^2} \left\{ \int_{-1}^1 \left(\frac{1+t}{1-t} \right)^{\mu} \frac{dt}{t-\cos x} + 2\lambda \int_{-1}^1 \left(\frac{1+t}{1-t} \right)^{\mu} \frac{dt}{1-2\lambda t + \lambda^2} \right\}.$$

If the formulae 2-11 and 2-12 are used we find without difficulty

$$k(x, \lambda) = \left(\frac{1+\lambda}{1-\lambda} \right)^{2\mu} \frac{1-\lambda^2}{1-2\lambda \cos x + \lambda^2}, \quad 7-29$$

from which

$$k_n(x) = e_n(2\mu) + 2 \sum_{k=1}^n e_k(2\mu) \cos(n-k)x. \quad 7-30$$

We may also write

$$k(x, \lambda) = \left(\frac{1+\lambda}{1-\lambda} \right)^{2\mu-1} \left\{ 1 + \cotg \frac{1}{2}x \frac{2\lambda \sin x}{1-2\lambda \cos x + \lambda^2} \right\}, \quad 7-31$$

so that by means of lemma 7-4

$$k_n(x) = c_n(2\mu-1) + 2 \cotg \frac{1}{2}x \sum_{k=1}^n c_k(2\mu-1) \sin(n-k)x. \quad 7-32$$

We shall next derive an estimate for $k_n(x)$. We shall prove the following property. There is a constant C not dependent on x such that

$$|k_n(x)| \leq \frac{C}{(\sin \frac{1}{2}x)^{2\mu} (\cos \frac{1}{2}x)^{1-2\mu}}, \quad 0 < x < \pi \quad 7-33$$

Proof For $k_n(x)$ we may write

$$k_n(x) = \frac{1}{2\pi i} \oint \left(\frac{1+w}{1-w} \right)^{2\mu} \frac{1-w^2}{1-2w \cos x + w^2} \frac{dw}{w^{n+1}},$$

where the path of integration is a small circle round $w=0$. The contour may be deformed as in section 4 and we get contributions from the residues at $w=e^{\pm ix}$ and the cuts from 1 to $+\infty$ and -1 to $-\infty$.

The residues give

$$\left(\frac{1+e^{ix}}{1-e^{ix}} \right)^{2\mu} e^{-nix} + \left(\frac{1+e^{-ix}}{1-e^{-ix}} \right)^{-2\mu} e^{nix} = O\left(\frac{1}{(\sin \frac{1}{2}x)^{2\mu}} \right).$$

The integral

$$\int_1^\infty \left(\frac{u+1}{u-1} \right)^{2\mu} \frac{u^2-1}{u^2-2u \cos x + 1} \frac{du}{u^{n+1}}$$

is less than

$$2^{1+2\mu} \int_0^\infty \frac{v^{1-2\mu} dv}{v^2+2(1-\cos x)(1+v)},$$

which in turn is less than

$$\frac{2}{(\sin \frac{1}{2}x)^{2\mu}} \int_0^\infty \frac{v^{1-2\mu} dv}{v^2+1},$$

thus the contribution from the positive real axis is also $O\left(\frac{1}{(\sin \frac{1}{2}x)^{2\mu}} \right)$.

In a similar way the contribution from the negative real axis is shown to be $O\left(\frac{1}{(\cos \frac{1}{2}x)^{1-2\mu}} \right)$.

If $\frac{\pi}{2} \leq x \leq \pi$ we may use formula 7-32 to get a better estimate near $x = \pi$. In section 4 we have derived the estimate $e_n(2\mu - 1) = O(n^{-2\mu})$. Thus we have also

$$|k_n(x)| \leq \frac{C'}{\sin \frac{1}{2}x} n^{1-2\mu}, \quad 0 < x \leq \pi \quad 7-34$$

We shall, however, be satisfied with the estimate

$$|k_n(x)| \leq \frac{C''}{(\sin \frac{1}{2}x)^{2\mu}} n^{1-2\mu}, \quad 0 < x \leq \pi \quad 7-35$$

for, using this, we find from lemma 7-5

$$\left| \sum_{n=1}^{\infty} r_n \sin nt k_n(x) \right| \leq C''' \frac{(\sin t)^{2\mu}}{(\sin \frac{1}{2}x)^{2\mu}},$$

and finally form $K(t, x)$

$$|K(t, x)| \leq C''' \sin^{2\mu} \frac{1}{2}t. \quad 7-36$$

Thus we have obtained the important result that the kernel $K(t, x)$ is uniformly bounded in the square $0 \leq t, x \leq \pi$.

§8 The orthogonal function

We shall again consider the expansion

$$f(x) = \sum_{n=1}^{\infty} a_n (\sin nx + \gamma_n \cos nx), \quad 8-1$$

where

$$\gamma_n = \gamma + r_n, \quad \operatorname{Re} \gamma > 0, \quad r_n = O\left(\frac{1}{n^2}\right).$$

The expansion

$$f(x) = \sum_{n=1}^{\infty} a_n^* \left(\sin nx + \frac{1}{\gamma_n} \cos nx \right), \quad 8-2$$

will be called the adjoint expansion. In particular we shall consider the adjoint expansion of unity

$$1 = \sum_{n=1}^{\infty} h_n \left(\sin nx + \frac{1}{\gamma_n} \cos nx \right). \quad 8-3$$

The problem of determining the coefficients h_n of 8-3 has been considered in the previous section. If now we introduce the function

$$h(x) = \sum_{n=1}^{\infty} h_n \sin nx = 1 - \sum_{n=1}^{\infty} \frac{h_n}{\gamma_n} \cos nx, \quad 0 < x < \pi \quad 8-4$$

it is clear that

$$\int_0^{\pi} h(x) \sin nx \, dx = \frac{\pi}{2} h_n, \quad n \geq 1 \quad 8-5$$

$$\int_0^{\pi} h(x) \cos nx \, dx = -\frac{\pi}{2} \frac{h_n}{\gamma_n}, \quad n \geq 1 \quad 8-6$$

From this we obtain

$$\int_0^{\pi} h(x) (\sin nx + \gamma_n \cos nx) \, dx = 0 \quad n \geq 1 \quad 8-7$$

which shows that the function $h(x)$ constructed in this way is orthogonal to all $\sin nx + \gamma_n \cos nx$. If γ_n is constant the formulae 8-3, 8-5, 8-6, 8-7 reduce to the formulae 4-9, 4-7, 4-8, 4-10 considered in section 4. If 8-1 converges uniformly we may apply 8-7 term by term and the following condition will be obtained

$$\int_0^{\pi} h(x) f(x) \, dx = 0. \quad 8-8$$

In a similar way as in the previous section $h(x)$ may be shown to satisfy a singular integral equation. We may either start from 8-3, apply the theory of the previous section or we may choose an independent argument. We prefer the latter and we start from

$$\lim_{N \rightarrow \infty} \int_0^{\pi} h(t) \sum_{n=1}^N (\cos nt + \frac{1}{\gamma_n} \sin nt) \cos nx \, dt = 0. \quad 8-9$$

Without difficulty we obtain

$$h(x) - \frac{1}{\gamma\pi} \int_0^{\pi} \frac{\sin t \, h(t) \, dt}{\cos t - \cos x} = g(x), \quad 8-10$$

with

$$g(x) = \frac{1}{\pi} \int_0^{\pi} K_3(t, x) h(t) \, dt + \frac{1}{\pi} \int_0^{\pi} h(t) \, dt, \quad 8-11$$

where

$$K_3(t, x) = 2 \sum_{n=1}^{\infty} \left(\frac{1}{\gamma} - \frac{1}{\gamma_n} \right) \sin nt \cos nx. \quad 8-12$$

Equation 8-10 is of the type 7-14 and will be solved in the same way. The solution of 8-10 becomes for known $g(x)$

$$h(x) = \sin^2 \mu \pi g(x) + \frac{\sin 2\mu\pi}{2\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} \int_0^{\pi} (\operatorname{tg} \frac{1}{2}u)^{-1+2\mu} \frac{\sin u \, g(u) \, du}{\cos u - \cos x}. \quad 8-13$$

If $g(x)$ is a constant, say $g(x)=1$, 8-13 gives

$$h(x) = \sin \mu \pi (tg \frac{1}{2}x)^{1-2\mu},$$

which is the orthogonal function for constant γ_n .

If for $g(x)$ we substitute 8-11 formula 8-10 reduces to

$$h(x) = \frac{\sin \mu \pi}{\pi} (tg \frac{1}{2}x)^{1-2\mu} \int_0^{\pi} h(t) dt + \frac{\sin \mu \pi}{\pi} \int_0^{\pi} K_4(t, x) h(t) dt, \quad 8-14$$

where

$$K_4(t, x) = \sin \mu \pi K_3(t, x) + \frac{\cos \mu \pi}{\pi} (tg \frac{1}{2}x)^{1-2\mu} \cdot \int_0^{\pi} \left(\frac{1+\cos u}{1-\cos u} \right)^{\frac{1}{2}-\mu} \frac{\sin u K_3(t, u) du}{\cos u - \cos x}. \quad 8-15$$

Finally we have as in the previous section the following result:

If we put

$$H(x) = (1+\cos x)^{\frac{1}{2}-\mu} h(x), \quad 8-16$$

$$H_0(x) = (1-\cos x)^{\frac{1}{2}-\mu} \frac{\sin \mu \pi}{\pi} \int_0^{\pi} h(t) dt, \quad 8-17$$

$$L(t, x) = (1-\cos x)^{\frac{1}{2}-\mu} (1+\cos t)^{-\frac{1}{2}+\mu} \sum_1^{\infty} \left(\frac{1}{\gamma} - \frac{1}{\gamma_n} \right) \sin nt k_n(x, \frac{1}{2}-\mu), \quad 8-18$$

where
$$\sum_0^{\infty} k_n(x, \frac{1}{2}-\mu) \lambda^n = \left(\frac{1+\lambda}{1-\lambda} \right)^{1-2\mu} \frac{1-\lambda^2}{1-2\lambda \cos x + \lambda^2},$$

$H(x)$ satisfies the ordinary Fredholm equation

$$H(x) = H_0(x) + \frac{\sin \mu \pi}{\pi} \int_0^{\pi} L(t, x) H(t) dt \quad 8-19$$

with a uniformly bounded kernel in the square $0 \leq t, x \leq \pi$.

§9 A potential problem

In this section we shall consider a potential problem which gives rise to the expansion of a given function into a trigonometrical series

$$f(x) = \sum_1^{\infty} a_n (\sin nx + \gamma_n \cos nx). \quad 9-1$$

Let the function $F(x,y)$ in the region $0 < x < \pi$, $y > 0$ satisfy the potential equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0, \quad 9-2$$

and the following boundary conditions

$$x = 0, \pi \quad F = 0, \quad 9-3$$

$$y \rightarrow \infty \quad F \rightarrow 0, \quad 9-4$$

$$y = 0 \quad \gamma \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} + \frac{1}{\pi} \int_{-\pi}^{\pi} R(x-\xi) \left(\frac{\partial F}{\partial x} \right)_{\xi} d\xi = f(x), \quad 9-5$$

where $\gamma > 0$, $f(x)$ and $R(x)$ are given functions, and where in particular

$$R(x) = \sum_1^{\infty} r_n \cos nx. \quad 9-6$$

If we represent $F(x,y)$ by

$$F(x,y) = \sum_1^{\infty} c_n \sin nx e^{-ny}, \quad 9-7$$

only the condition 9-5 has to be considered.

Since

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos m(x-\xi) \cos n\xi d\xi = \delta_{mn} \cos nx$$

we get

$$f(x) = \sum_1^{\infty} nc_n (\sin nx + \gamma_n \cos nx), \quad 9-8$$

where

$$\gamma_n = \gamma + r_n.$$

If $a_n = nc_n$ this is exactly the representation 9-1.

Thus the solution of the potential problem considered above may be reduced to the solution of the representation problem 9-1.

In the preceding sections it has been shown that the representation problem 8-1 may be reduced to a singular integral equation. We shall now show that this singular integral equation also can be derived from the potential problem by means of Green's theorem.

If $R(x) \equiv 0$ the coefficients c_n may be determined by explicit analytic expressions. We have seen that for $n \rightarrow \infty$

$$c_n = O(n^{-2+2\mu}), \quad \mu\pi = \arctan \gamma.$$

This means that both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ become infinite at $x = \pi$. For $0 \leq x < \pi$, however, they are bounded. In particular at $x = \pi$ $F_x, F_y = O\{(\pi-x)^{-2\mu}\}$. If $f(x)$ satisfies the condition

$$\int_0^{\pi} (\operatorname{tg} \frac{1}{2}x)^{1-2\mu} f(x) dx = 0$$

F_x and F_y are uniformly bounded in the closed interval $(0, \pi)$. If $R(x) \neq 0$ and if $r_n = O(\frac{1}{n^2})$ similar conclusions may be drawn.

Let $G(x, y, \xi, \eta)$ be a Green function in the region $0 < x < \pi, y > 0$ and let

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0$$

$$x = 0, \pi \quad G = 0$$

$$y \rightarrow \infty \quad G \rightarrow 0$$

$$(x-\xi)^2 + (y-\eta)^2 \rightarrow 0 \quad G \sim \frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

then applying Green's theorem on the region considered above one easily obtains

$$F(\xi, \eta) = \int_0^{\pi} G(x, 0, \xi, \eta) \frac{\partial}{\partial y} F(x, 0) dx - \int_0^{\pi} \left\{ \frac{\partial}{\partial y} G(x, 0, \xi, \eta) \right\} F(x, 0) dx \quad 9-9$$

Taking $\eta = 0$ an integral equation is obtained between the partial derivatives of F at zero for which also the relation 9-5 exists.

Let $G_0(x, y, \xi, \eta)$ represent the standard Green function of the full strip $0 < x < \pi$. Obviously we have

$$G_0(x, y, \xi, \eta) = \frac{1}{2\pi} \operatorname{Re} \left\{ \ln \sin \frac{(x-\xi)+i(y-\eta)}{2} - \ln \sin \frac{(x+\xi)+i(y-\eta)}{2} \right\} \quad 9-10$$

Simplification may be expected if in 9-9 the following Green functions are chosen

$$G_1 = G_0(x, y, \xi, \eta) - G_0(x, y, \xi, -\eta) \quad 9-11$$

$$G_2 = G_0(x, y, \xi, \eta) + G_0(x, y, \xi, -\eta) \quad 9-12$$

so that at $y=0$ $G_1=0$ and $\frac{\partial G_2}{\partial y} = 0$.

We note the following expressions

$$G_0(x, y, \xi, \eta) = \frac{1}{4\pi} \ln \frac{\sin^2 \frac{x-\xi}{2} + \operatorname{sh}^2 \frac{y-\eta}{2}}{\sin^2 \frac{x+\xi}{2} + \operatorname{sh}^2 \frac{y+\eta}{2}}$$

$$2\pi \frac{\partial}{\partial x} G_0(x, 0, \xi, 0) = \frac{\sin \xi}{\cos \xi - \cos x}$$

$$2\pi \frac{\partial}{\partial \xi} G_0(x, 0, \xi, 0) = \frac{\sin x}{\cos x - \cos \xi}$$

$$\frac{\partial G_1}{\partial \eta} + \frac{\partial G_2}{\partial y} = 0.$$

If in 9-9 we substitute G_1 the first integral on the right-hand side vanishes. After a differentiation with respect to η we get

$$\frac{\partial F(\xi, \eta)}{\partial \eta} = - \int_0^\pi \frac{\partial^2 G_1}{\partial y \partial \eta} F(x, 0) dx$$

but $-\frac{\partial^2 G_1}{\partial y \partial \eta} = \frac{\partial^2 G_2}{\partial y^2} = -\frac{\partial^2 G_2}{\partial x^2}$, so that after a partial integration and change of notation

$$y=0 \quad \left(\frac{\partial F}{\partial y}\right)_x + \frac{1}{\pi} \int_0^\pi \left(\frac{\partial F}{\partial x}\right)_t \frac{\sin x dt}{\cos t - \cos x} = 0 \quad 9-13$$

From 9-5 and 9-13 the following singular integral equation for $\frac{\partial F}{\partial x}$ is obtained

$$y=0 \quad \gamma \frac{\partial F}{\partial x} + \frac{1}{\pi} \int_0^\pi \left(\frac{\partial F}{\partial x}\right)_t \frac{\sin x dt}{\cos t - \cos x} + \frac{1}{\pi} \int_{-\pi}^\pi R(x-t) \left(\frac{\partial F}{\partial x}\right)_t dt = f(x) \quad 9-14$$

This integral equation is clearly equivalent with that obtained in section 7 formula 7-8.

If in 9-9 G_2 is substituted we obtain at once

$$\frac{\partial F(\xi, \eta)}{\partial \xi} = \int_0^\pi \frac{\partial G_2}{\partial \xi} \left(\frac{\partial F}{\partial y}\right)_x dx$$

or at $y=0$

$$y=0 \quad \left(\frac{\partial F}{\partial x}\right)_x - \frac{1}{\pi} \int_0^\pi \left(\frac{\partial F}{\partial y}\right)_t \frac{\sin t dt}{\cos t - \cos x} = 0. \quad 9-15$$

If again in 9-15 $\frac{\partial F}{\partial y}$ is expressed by means of 9-5 another singular integral equation is obtained which turns out to be equivalent to that of formula 7-10.

§10 Conclusions

The problem of expanding a function $f(x)$, which is hölderian in the interval $0 \leq x \leq \pi$, into a trigonometrical series of the following kind

$$f(x) = \sum_{n=1}^{\infty} a_n (\sin nx + \gamma_n \cos nx) \quad 10-1$$

has been solved completely for the case of $\gamma_n = \gamma$ which is not dependent on n .

In the case of γ_n variable, however, we have only shown that, if

$$\gamma_n = \gamma + O(n^{-2}) \quad 10-2$$

the problem considered above is equivalent to an ordinary Fredholm integral equation with bounded kernel. Thus in a particular case of a given function $f(x)$ and numerically given phase factors γ_n the coefficients a_n may be determined by any numerical process by which the solution of a Fredholm equation is obtained. The discussion of the Fredholm equation will be given in another report, but we remark already here that there might be exceptional cases of sets $\{\gamma_n\}$ satisfying 10-2 for which the problem 10-1 does not admit a solution or admits an infinity of solutions. Those cases correspond to the proper values of the Fredholm equation.

A similar remark may be made in connection with the function $h(x)$ which is orthogonal to the functions $\sin nx + \gamma_n \cos nx$, $n \geq 1$. Also the discussion of the numerical determination of $h(x)$, if existing, will be given later on.

If the assumption 10-2 is dropped the problem 10-1 becomes much more difficult, and apart from the following case $f(x) \in L_2$, $|\gamma_n| < \gamma < 1$ considered by Paley and Wiener almost nothing seems to be known about the possibility of the expansion 10-1.
